What are high-dimensional permutation? How many are there?

Nati Linial and Zur Luria

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Today we want to seek high-dimensional counterparts of permutations.
It is a recurring theme that in moving to higher dimensions many simple, even trivial facts that we are very used to, take on a new life and become richer and more geometric.

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Recall: an elementary collapse is a step in which we remove the unique edge that touches a leaf vertex. We say that G is collapsible provided that all its edges can be eliminated through a series of elementary collapses.

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By counting dimensions the two conditions are equivalent for an n-vertex graph with $n-1$ edges.

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A high-dimensional perspective of trees (contd.)

In the d-dimensional case we consider the inclusion matrix M of the $(d-1)$ vs. d-dimensional faces of the simplicial complex X. (Viewed as a linear operator, this matrix is the boundary operator ∂_d.) Well, actually, we should be talking about the signed inclusion matrix to account for orientation, but let's ignore it. Alternatively we can work over $\mathbb{F}_2$ to do away with the signs.

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If the simplicial complex X is d-dimensional, has n vertices, a full $(d-1)$-dimensional skeleton and $\binom{n-1}{d}$ d-faces, then again by couting dimensions, the two conditions are equivalent.

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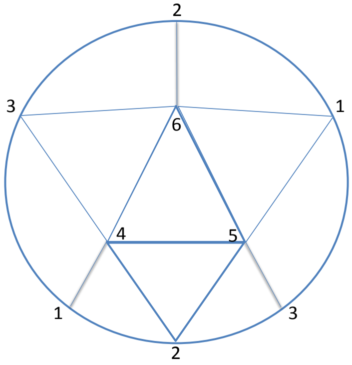
The implication collapsibility $\to$ vanishing of the homology holds in all dimensions and over every base field.

However, in higher dimension these conditions are no longer equivalent

In particular, the underlying field cannot be ignored.



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A line here means either a row or a column.

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Whereas a matrix has two kinds of lines, namely rows and columns, now there are $d + 1$ kinds of lines.
A line is a set of n entries in the array that are obtained by fixing d out of the $d + 1$ coordinates and the letting the remaining coordinate take all values from 1 to n.

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The case $d = 2$. Don't I know you from somewhere?

According to our definition, a 2-dimensional permutation on $[n]$ is an $[n] \times [n] \times [n]$ array of zeros and ones in which every row every column and every shaft contains exactly one 1-entry.
An equivalent description can be achieved by using a topographical map of this terrain.

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It is easily verified that the defining condition is that in this array every line (row or column) contains every entry $n \geq i \geq 1$ exactly once.

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It is easily verified that the defining condition is that in this array every line (row or column) contains every entry $n \geq i \geq 1$ exactly once.
In other words: Two-dimensional permutations are synonymous with Latin Squares.

Where do we go from here

There are so many things we know about ("one-dimensional") permutations. Let's see if we can develop the analogous high-dimensional theory. We know

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- We know what they look like typically.
- Birkhoff von-Neumann Thm on doubly stochastic matrices.
- Even trivial properties can turn into interesting questions in higher dimension.

The count - An interesting numerology

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As we will discuss below the count of order-n Latin squares is

$$|\mathcal{L}_{\mathrm{n}}| = \left((1+\mathrm{o}(1))\frac{\mathrm{n}}{\mathrm{e}^2}\right)^{\mathrm{n}^2}$$

So, let us conecture

Conjecture

The number of d-dimensional permutations on [n] is

$$|S_n^d| = \left((1+o(1))\frac{n}{e^{d}}\right)^{n^{d}}$$

and what we actually know

At present we can only prove the upper bound

Theorem

The number of d-dimensional permutations on $[n]$ is

$$|S_n^d| \leq \left((1+o(1))\frac{n}{e^d}\right)^{n^d}$$

How do you prove the estimate for the number of Latin Squares?

Recall that the permanent of a square matrix is a "determinant without signs".

$$per(A) = \sum_{\sigma \in S_n} \prod a_{i,\sigma(i)}$$

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- It counts perfect matchings in bipartite graphs.
- In other words, it counts the generalized diagonals included in a $0/1$ matrix.
- It is $\#$-P-hard to calculate the permanent exactly, even for a $0/1$ matrix.
- On the other hand there is an efficient approximation scheme for permanents of nonnegative matrices.

A lower bound on the permanent

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By the marriage theorem, a doubly stochastic matrix has a positive permanent. The set of doubly stochastic matrices is a convex polytope. The permanent is a continuous function, so: What is min $per\ A$ over $n \times n$ doubly-stochastic matrices? As conjectured by van der Waerden in the 20's and proved over 50 years later by Falikman and by Egorichev, in the minimizing matrix all entries are $\frac{1}{n}$.

Theorem

The permanent of every $n \times n$ doubly stochastic matrix is $\geq \frac{n!}{n^n}$.

An upper bound on permanents

The following was conjectured by Minc and proved by Brégman

Theorem

Let A be an $n \times n$ $0/1$ matrix with r_i ones in the i-th row $i = 1, \ldots, n$. Then $perA \leq \prod_i (r_i!)^{1/r_i}$. The bound is tight.

Our work

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What about a matching lower bound?
We don't have it (yet....), but there is a reason.

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It is an easy consequence of the marriage theorem that if in a $0/1$ matrix A, all row sums and all column sums equal $k \geq 1$, then $perA > 0$.
The analogous statement is no longer true in higher dimensions.
Here is an example of a $4 \times 4 \times 4$ array with two zeros and two ones in every line which contains no 2-permutation.

An example

$$\begin{matrix} * & 0 & * & 0 \\ 0 & 1 & 0 & 1 \\ * & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{matrix} \qquad \begin{matrix} 0 & * & 0 & * \\ 0 & 0 & 1 & 1 \\ * & * & 0 & 0 \\ 1 & 0 & 1 & 0 \end{matrix}$$

$$\begin{matrix} 0 & 0 & * & * \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{matrix}$$

Approximately counting Latin squares

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Namely, A is an $n \times n \times n$ array of $0/1$ where every line has a single 1 entry.
Note that every layer in A is a permutation matrix.
Given several layers in A, how many permutation matrices can play the role of the next layer?

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Let B be a $0/1$ matrix where $b_{ij} = 1$ iff in all previous layers the ij entry is zero.

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The estimate for the number of Latin squares is attained by bounding at each step the number of possiblities for the next layer
from below, using the van der Waerden's conjecture) and from above, using Minc-Brégman.

Back to basics - Reproving Brégman's theorem

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One of the insights gained about Brégman's theorem is that it is useful to interpret it using the notion of entropy.
So let us review the basics of this method.

A quick recap of entropy

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$$H(X) = -\sum p_i \log p_i$$

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All logarithms here are to base e. This is not the convention when it comes to entropy, but it will make things more convenient for us.

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If X and Y are two discrete random variables, then the conditional entropy

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The chain rule is one of the fundamental properties of entropy: If $X_1, \ldots, X_n$ are discrete random variables defined on the same probability space, then

$$H(X_1, \ldots, X_n) = H(X_1) + H(X_2|X_1) + H(X_3|X_1, X_2) + \ldots$$

Proving Brégman's theorem using entropy

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$$H(X) = \log(perA).$$

Therefore, an upper bound on $H(X)$ yields an upper bound on $perA$, which is what we want.

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We next express $X = (X_1, \ldots, X_n)$, where X_i is the index of the single 1-entry that is selected by the generalized diagonal X at the i-th row. How should we interpret the relation

$$H(X) = H(X_1) + H(X_2|X_1) + H(X_3|X_1, X_2) + \ldots$$

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In particular, what can we say about $H(X_i|X_1, \ldots, X_{i-1})$?

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If N_i is the (random) number of 1-entries in the i-th row that remain unshaded when the i-th row is reached, then clearly

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Very nice. The troube is that we know very little about the random variable N_i.

A good trick

The way around this difficulty is not to sum the terms $H(X_i | X_1, \ldots, X_{i-1})$ in the normal order, but rather introduce σ, a random ordering of the rows.

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What can we say about the expectation of $\log N_i^\sigma$?

This can be restated as follows: You are expecting r visitors who arrive at random, independently chosen times.

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One of the visitors is your guest of honor and you are interested in his (random) arrival rank among the r visitors.

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$$\mathbb{E}(\log N) = \frac{1}{r} \sum_{j=1,\dots,r} \log j = \frac{\log r!}{r} = \log(r!)^{1/r}.$$

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By summing over all rows, the Brégman bound is established

$$H(X) = \log(perA) \leq \sum_i \log(r_i!)^{1/r_i}.$$

Doing the d-dimensional case

If A is an $n^{d+1} = n \times n \times \ldots \times n$ array of $0/1$ we define $per_d(A)$ to be the number of d-permutations that are included in A. Let's consider all lines in A in the same direction, say lines of the form $l_{\mathbf{i}} = (i_1, \ldots, i_d, *)$. Let $r_{\mathbf{i}}$ be the number of 1's in the line $l_{\mathbf{i}}$.

Theorem

Let A be an $[n]^{d+1}$ *array of* $0/1$, *and let* $r_{\mathbf{i}}$ *be the number of* 1*'s in the line* $l_{\mathbf{i}}$, *as above. Then*

$$per_d(A) \leq \prod_{\mathbf{i}} \exp(f(d, r_{\mathbf{i}})).$$

The function $f(d, r)$ *is defined recursively, via* $f(0, r) = \log r$, *and*

$$f(d, r) = \frac{1}{r} \sum_{k=1,\ldots,r} f(d-1, k).$$

A few words about the proof

The general strategy remains the same, using entropy. The random variable X takes uniformly d-permutations that are contained in A.

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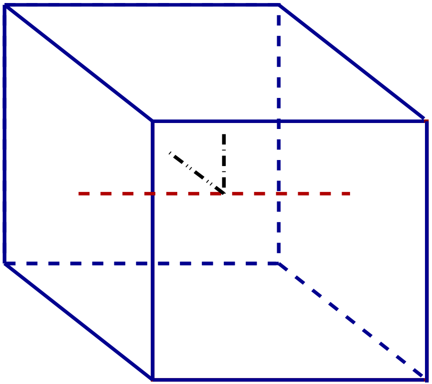
The general strategy remains the same, using entropy. The random variable X takes uniformly d-permutations that are contained in A. The main new ingredient in the proof is the choice of the random ordering σ. The expression

$$f(d,r) = \frac{1}{r} \sum_{k=1,\dots,r} f(d-1,k).$$

suggests that the story with the randomly-arriving guests is modified as follows:

Different shades, randomly-arriving visitors



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The visitors are arriving at a random order. Only the guest of honor and everyone who came after him remain in the game and are invited again.
This procedure is repeated d times and we ask about N, the (random) number of guests who never showed up (in all d repeats) before the guest of honor.

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In the shade terminology, we first eliminate the ones in row $\mathbf{i}$ that are shaded in direction $d + 1$.
(This is the first round of invitation). Shades that come from other directions correspond to our subsequent rounds of inviting the guests.

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Stirling's formula says that the geometric mean of these numbers is $(1 + o(1))\frac{r}{e}$.
In other words, the arithmetic mean of the numbers $\log j$ over $j = 1, \ldots, r$ is $\log r - 1$.
When we repeat the above process of inviting guests and re-inviting only the late arrivals (what a strange idea???) we lose this 1 term d times.

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While we find this definition very appealing and the open questions are fascinating, this is by no means the final word on the subject.

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An acyclic high-dimensional tournament seems like another good analog of a permutation.
There is still too little to report from that front.

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- Similar questions for related concepts: tournaments, STS, 1-factorizations ...
- Efficient random generation of such objects.
- Investigating the typical and extremal properties of d-dimensional permutations.

That's all folks